CHAPTER 2: CAPITAL MARKET IMPERFECTIONS

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Key Result from Chapter 1

- Along a perfect foresight equilibrium path, a small open economy will smooth out consumption over time regardless of the output path.
- 3 key assumptions:
  1) The economy has perfect access to international capital markets.
  2) The output path is known with certainty.
  3) The country is able to precommit to repaying its debt.
- This chapter analyzes the consequences of relaxing each of the above assumptions.

2. Financial Autarky

- Autarky is the quality of being self-sufficient. Usually the term is applied to political states or their economic policies.
- Autarky exists whenever an entity can survive or continue its activities without external assistance.
- Autarky is not necessarily economic.
- By financial autarky we will mean that the economy as a whole cannot borrow from the rest of the world.

Set-up of the Model

- Small open economy inhabited by a large number of identical, two-period-lived consumers.
- No uncertainty, perfect foresight.
- Only one good (tradable and non-storable).
- Small open economy: takes price of tradable good as given.
- The economy is endowed with a flow of the good (*i.e.* no production).
- No government.
- ~~Perfect capital mobility in the sense that consumers can borrow/lend in international capital markets at a constant real interest rate, *r*.~~ Financial autarky.

2.1 Consumer's Problem
2.1.1 Budget Constraints

Let b_i^d, $i = 1,2$ stand for the level of net domestic assets at the end of period i. The flow budget constraint for period 1 is thus given by :

$$b_1^d = y_1 - c_1 \qquad (1)$$

while $b_2^d \geq 0$

non-satiation implies that this holds with equality. Then the flow budget constraint for period 2 is

$$c_2 = (1+\rho)b_1^d + y_2 \qquad (2)$$

where ρ is the domestic real interest rate. Given that the domestic bond is non-tradable, ρ may differ from the international real interest rate, r. Combining (1) and (2) yields the consumer's intertemporal budget constraint :

$$y_1 + \frac{y_2}{1+\rho} = c_1 + \frac{c_2}{1+\rho} \qquad (3)$$

2.1.2 Utility Maximization

The lifetime utility of the representative household is given by :

$$W = u(c_1) + \beta u(c_2) \qquad (4)$$

The maximization problem is to choose c_1 and c_2 (note that we are implicitly deciding b_1^d as we choose c_1) to :

$$\max_{\{c_1, c_2\}} W = u(c_1) + \beta u(c_2)$$

subject to

$$y_1 + \frac{y_2}{1+\rho} = c_1 + \frac{c_2}{1+\rho}$$

Utility Maximization

$$\max_{\{c_1,c_2\}} L = u(c_1) + \beta u(c_2) + \lambda\left[y_1 + \frac{y_2}{1+\rho} - c_1 - \frac{c_2}{1+\rho}\right]$$

F.O.N.C.:

$$\left.\begin{aligned} u'(c_1) &= \lambda \\ \beta u'(c_2) &= \frac{\lambda}{1+\rho} \end{aligned}\right\} \quad u'(c_1) = \beta(1+\rho)u'(c_2) \qquad (7)$$

If the economy has full access to international capital markets, $\rho = r$, which given the assumption that $\beta = \frac{1}{1+r}$ implies

$$u'(c_1) = u'(c_2) \qquad (8)$$

2.2 Equilibrium Conditions

2.2.1 Bond market equilibrium : $$b_1^d = 0 \quad (9)$$

2.2.2 Aggregate constraints : $$c_1 = y_1 \quad (10)$$

$$c_2 = y_2 \quad (11)$$

$$TB_i = CA_i = 0 \quad \text{for } i = 1,\ 2$$

2.3 Solution of the Model

We have 3 endogenous variables: c_1, c_2 and ρ.

Consumption levels are determined by $c_1 = y_1$ (10) and $c_2 = y_2$ (11).

Meanwhile, the domestic interest rate can be obtained from the Euler equation (7) to get

$$1+\rho = \frac{u'(y_1)}{\beta u'(y_2)} \qquad (12)$$

$\therefore$ in financial autarky, the domestic interest rate, ρ, is solely determined by the path of output.

2.4 Stationary Equilibrium

Assume output path is flat: $y_1 = y_2 = y$

It then follows from (10) and (11) that: $c_1 = c_2 = y$

$$\left(\bar{c} = \frac{1+r}{2+r}\left[y + \frac{y}{1+r} \right] = y, \quad b_1^{d*} = 0 \right)$$

From (12) it follows that: $1+\rho = \dfrac{1}{\beta}$

∴ Faced with a constant output path, households have no incentive whatsoever to save or dissave. Hence, being shut off from international capital markets is not a binding constraint.

2.5 Non-stationary Equilibrium

Now suppose $y_1 < y_2$

From (10) and (11) it follows that $y_1 = c_1 < c_2 = y_2$

Since $c_1 < c_2$ it follows from (12) that $1+\rho > \frac{1}{\beta}$ as $u'(y_1) > u'(y_2)$

Under our maintaned assumption that $\beta = \frac{1}{1+r}$ we have

$$1+\rho > \tfrac{1}{\beta} = 1+r \quad \Rightarrow \quad \rho > r$$

i.e. the domestic real interest rate is higher than the international one.

Welfare Consequences of Limited Credit Access

Now, to show that financial autarky is costly since welfare would be higher if the economy were able to smooth consumption over time ($i.e., c_1 = c_2 = \bar{c}$) use the budget constraint

$$\bar{c} + \frac{\bar{c}}{1+r} = y_1 + \frac{y_2}{1+r} \quad \Rightarrow \quad \bar{c} = \frac{1+r}{2+r}\left[y_1 + \frac{y_2}{1+r}\right] \qquad (13)$$

Welfare under perfect capital mobility would thus be higher than under financial autarky if

$$\underbrace{u(\bar{c}) + \beta u(\bar{c})}_{\substack{\text{welfare under} \\ \text{perfect capital mobility}}} > \underbrace{u(y_1) + \beta u(y_2)}_{\substack{\text{welfare under} \\ \text{financial autarky}}} \qquad (14)$$

Welfare Consequences of Limited Credit Access

This condition for perfect capital mobility to dominate financial autarky can be reexpressed as :

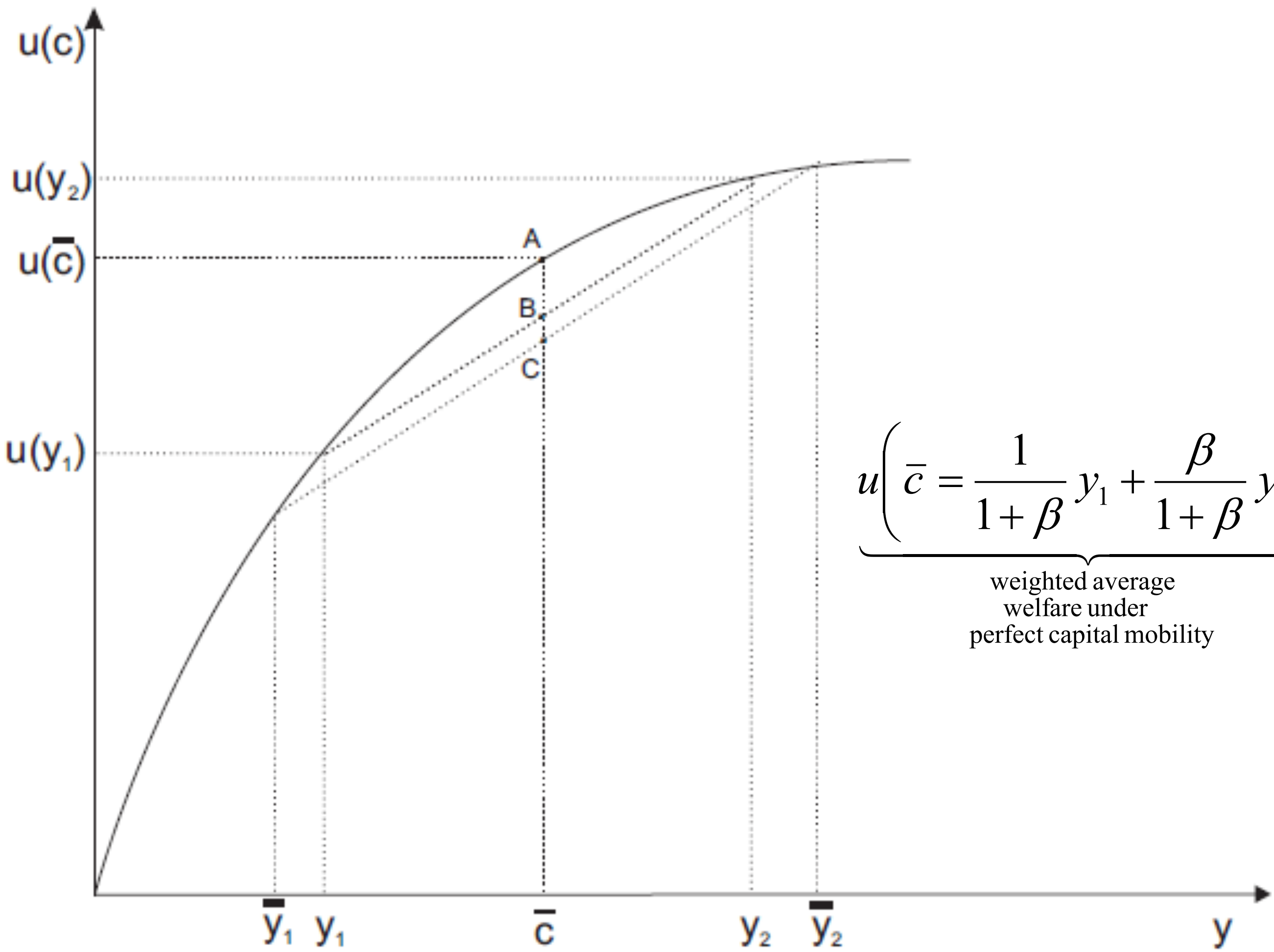
$$u(\bar{c}) > \frac{1}{1+\beta}u(y_1) + \frac{\beta}{1+\beta}u(y_2) \qquad (15)$$

Since by assumption $\beta = \frac{1}{1+r} \Rightarrow 1+r = \frac{1}{\beta} \Rightarrow 2+r = \frac{1+\beta}{\beta}$ then

$$\bar{c} = \frac{1+r}{2+r}\left[y_1 + \frac{y_2}{1+r}\right] = \frac{\frac{1}{\beta}}{\frac{1+\beta}{\beta}}\left[y_1 + \beta y_2\right] = \frac{1}{1+\beta}y_1 + \frac{\beta}{1+\beta}y_2$$

Hence, by strict concavity of $u(\cdot)$, inequality (15) holds for any pair of values $y_1 \neq y_2$.

Welfare Consequences of Limited Credit Access



$$\underbrace{u\left(\bar{c} = \frac{1}{1+\beta} y_1 + \frac{\beta}{1+\beta} y_2\right)}_{\substack{\text{weighted average} \\ \text{welfare under} \\ \text{perfect capital mobility}}} > \underbrace{\frac{1}{1+\beta} u(y_1) + \frac{\beta}{1+\beta} u(y_2)}_{\substack{\text{weighted average} \\ \text{welfare under} \\ \text{autarky}}}$$

Comparison Financial Autarky versus Perfect Capital Mobility

- In chapter 1, under perfect capital mobility, we learned that a temporary lower level of output results in a trade balance worsening but (almost) no change in consumption.

- Now, under financial autarky, a temporarily lower level of output will lead to reduce consumption, a higher real interest rate and no change in the trade balance.

3. Uncertain Output Path

- Suppose now that households have access to international capital markets but they did not know for sure what their future endowment will be.
- How would they decide whether to save or disave?
- Now y_1 is known with certainty, while y_2 can take two possible values:

$$y_2 = \begin{cases} y_2^H \ with\ probability & p \\ y_2^L \ with\ probability & 1-p \end{cases} \quad (16)$$

  - To induce a desire to smooth consumption assume that:

$$y_1 < E\{y_2\} = p y_2^H + (1-p) y_2^L \quad (17)$$

Set-up of the Model

- Small open economy inhabited by a large number of identical, two-period-lived consumers.
- ~~No uncertainty, perfect foresight.~~ Uncertainty about second period endowment.
- Only one good (tradable and non-storable).
- Small open economy: takes price of tradable good as given.
- The economy is endowed with a flow of the good (*i.e.* no production).
- No government.
- Perfect capital mobility in the sense that consumers can borrow/lend in international capital markets at a constant real interest rate, *r*.

3.1 Incomplete Markets

The key feature of incomplete contracts is that net foreign assets (or debt obli-gations), b_1, are not contingent on the realization of the uncertain output y_2.

3.1.1 Budget Constraints

For simplicity assume that $b_0 = 0$. Then in period 1

$$b_1 = y_1 - c_1 \tag{18}$$

while now we have two constraints for period 2, one for each realization of y_2 :

$$c_2^H = (1+r)b_1 + y_2^H \tag{19}$$

and

$$c_2^L = (1+r)b_1 + y_2^L \tag{20}$$

Budget Constraints (cont.)

Combining : $b_1 = y_1 - c_1$ (18) with

$$c_2^H = (1+r)b_1 + y_2^H \quad (19) \quad \text{and} \quad c_2^L = (1+r)b_1 + y_2^L \quad (20)$$

we obtain two intertemporal budget constraints, one for each of the two possible output paths :

$$y_1 + \frac{y_2^H}{1+r} = c_1 + \frac{c_2^H}{1+r} \qquad (21)$$

and

$$y_1 + \frac{y_2^L}{1+r} = c_1 + \frac{c_2^L}{1+r} \qquad (22)$$

Budget Constraints (cont.)

Multiplying the intertemporal budget constraint for each state $\{H, L\}$, given by equations (21) and (22), by their probabilities $\{p, 1-p\}$ and adding them, we get the intertemporal budget constraint in expected values:

$$p\left[y_1 + \frac{y_2^H}{1+r} = c_1 + \frac{c_2^H}{1+r}\right]$$

$$+ \quad (1-p)\left[y_1 + \frac{y_2^L}{1+r} = c_1 + \frac{c_2^L}{1+r}\right]$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$y_1 + \frac{py_2^H + (1-p)y_2^L}{1+r} = c_1 + \frac{pc_2^H + (1-p)c_2^L}{1+r}$$

or

$$y_1 + \frac{E\{y_2\}}{1+r} = c_1 + \frac{E\{c_2\}}{1+r} \qquad (31)$$

3.1.2 Utility Maximization

Given the second period output´s uncertainty, now we have an expected life - time utility :

$$W = u(c_1) + \beta E\{u(c_2)\} \quad (23)$$

which can be reexpressed taking into account the distribution of output as :

$$W = u(c_1) + \beta\left[pu(c_2^H) + (1-p)u(c_2^L)\right] \quad (24)$$

As before, the maximization problem can be done either :

(a) using the three period by period constraints (18), (19) and (20), or

(b) using the two life - time resource constraints (21) and (22).

Given that we already know that this election does not affect the results, lets do it using the latter approach adding the multipliers λ^H and λ^L.

Utility Maximization (cont.)

$$\max_{\{c_1, c_2^H, c_2^L, \lambda^H, \lambda^L\}} L = u(c_1) + \beta\left[pu(c_2^H) + (1-p)u(c_2^L)\right]$$

$$+\lambda^H\left[y_1 + \frac{y_2^H}{1+r} - c_1 - \frac{c_2^H}{1+r}\right] + \lambda^L\left[y_1 + \frac{y_2^L}{1+r} - c_1 - \frac{c_2^L}{1+r}\right] \qquad (25)$$

The first - order necessary conditions for a maximum include :

$$\frac{\partial L}{\partial c_1} = u'(c_1) - \lambda^H - \lambda^L = 0$$

$$\frac{\partial L}{\partial c_2^H} = \beta p u'(c_2^H) - \frac{\lambda^H}{1+r} = 0$$

$$\frac{\partial L}{\partial c_2^L} = \beta(1-p)u'(c_2^L) - \frac{\lambda^L}{1+r} = 0$$

Combining these constraints you can get the (stochastic) Euler equation

$$u'(c_1) = \beta(1+r)\left[pu'(c_2^H) + (1-p)u'(c_2^L)\right] \qquad (26)$$

or

$$u'(c_1) = \beta(1+r)E\{u'(c_2)\} \qquad (27)$$

3.2 Equilibrium

We have the Euler equation : $u'(c_1) = \beta(1+r)E\{u'(c_2)\}$.

Assuming $\beta = \frac{1}{1+r}$ we get that marginal utility of consumption in **expected value** is equalized across periods : $u'(c_1) = E\{u'(c_2)\}$.

The solution of the model will depend on the relationship between the expected marginal utility $E\{u'(c_2)\}$ of the Euler equation and the marginal utility of expected consumption $u'(E\{c_2\})$.

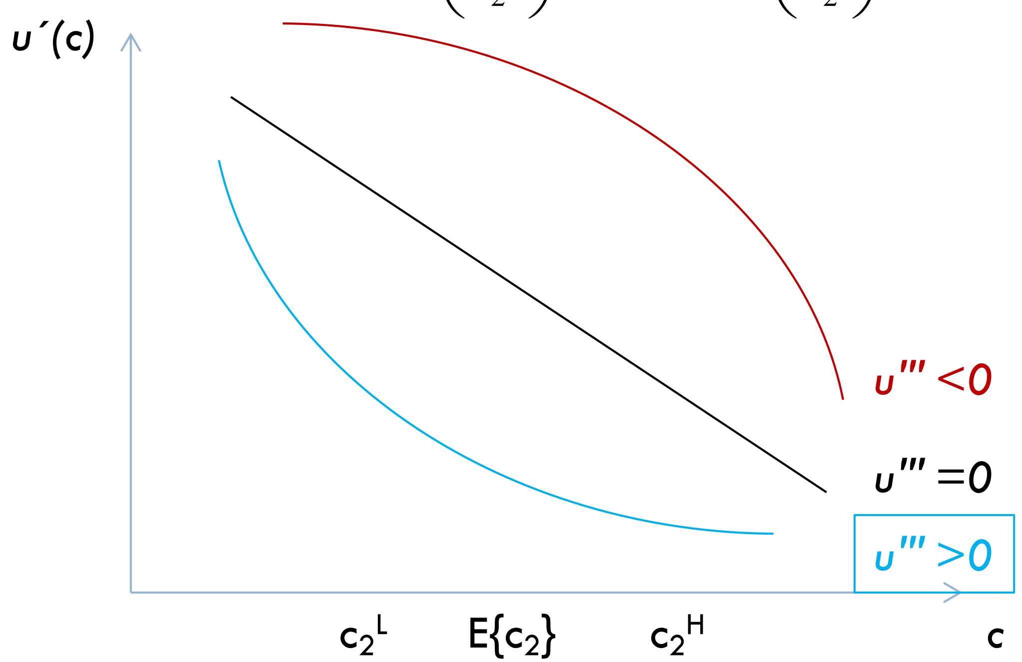
If $E\{u'(c_2)\} = u'(E\{c_2\}) \Rightarrow c_1 = E\{c_2\}$ certainty equivalence (28)

If $E\{u'(c_2)\} > u'(E\{c_2\}) \Rightarrow c_1 < E\{c_2\}$ precautionary savings (29)

If $E\{u'(c_2)\} < u'(E\{c_2\}) \Rightarrow c_1 > E\{c_2\}$ imprudence (30)

Equilibrium (cont.)

- If $u''' = 0 \Rightarrow u'$ is linear $\Rightarrow pu'\left(c_2^H\right) + (1-p)u'\left(c_2^L\right) \equiv E\{u'(c_2)\} = u'(E\{c_2\})$
- If $u''' > 0 \Rightarrow u'$ is strictyly convex $\Rightarrow pu'\left(c_2^H\right) + (1-p)u'\left(c_2^L\right) \equiv E\{u'(c_2)\} > u'(E\{c_2\})$
- If $u''' < 0 \Rightarrow u'$ is strictyly concave $\Rightarrow pu'\left(c_2^H\right) + (1-p)u'\left(c_2^L\right) \equiv E\{u'(c_2)\} < u'(E\{c_2\})$



Case 1: certainty equivalence $u'''(c)=0$

When we have quadratic preferences $u''' = 0$ and u' is linear.

In this case:

$$u'(c_1) = E\{u'(c_2)\} = u'(E\{c_2\})$$

Since u' is a strictly decreasing function (recall that $u'' < 0$) it follows that:

$$c_1 = E\{c_2\} \qquad (32)$$

In an expected value sense, therefore, the economy smooths consumption over time. This is Robert Hall´s (1978) celebrated result that under quadratic preferences and $\beta = \frac{1}{1+r}$ consumption follows a random walk.

Case 1 (cont.)

This does not mean that actual consumption is smoothed, i.e. $c_1 = c_2$.

To compute a reduced form for consumption, combine (32) with (31) to obtain :

$$c_1 = \frac{1+r}{2+r}\left[y_1 + \frac{E\{y_2\}}{1+r} \right] \qquad (33)$$

Therefore, c_1 is only a function of $E\{y_2\}$ and not of its variance.

To derive CA_1 $\left(= TB_1 \text{ as } b_0 = 0 \text{ by assumption}\right)$ combine (18) and (33) to get

$$CA_1 = b_1 - b_0 = y_1 - c_1 = \frac{1}{2+r}\left[y_1 - E\{y_2\} \right] < 0 \qquad CA \text{ deficit}$$

Case 1 (cont.)

Substituting b_1 into (19) and (20), we obtain reduced forms for c_2^H and c_2^L :

$$c_2^H = y_2^H + \frac{1+r}{2+r}\left[y_1 - E\{y_2\}\right] < y_2^H \qquad (34)$$

$$c_2^L = y_2^L + \frac{1+r}{2+r}\left[y_1 - E\{y_2\}\right] < y_2^L \qquad (35)$$

Therefore, $c_2^i < y_2^i$ for $i = H, L$ as $b_1 < 0$, i.e. debt needs to be repaid. Also $c_2^i = f\left(y_2^i\right)$ and $c_2^H > c_2^L$. Incomplete markets thus introduce a positive correlation between c_2 and output. In fact, from (34) and (35) it follows that :

$$E\{c_2\} = E\{y_2\} + \frac{1+r}{2+r}\left[y_1 - E\{y_2\}\right] \qquad (36)$$

$$Var\{c_2\} = Var\{y_2\} \qquad (37)$$

Case 1 (cont.)

Finally notice that

$$\underbrace{u(c_1)+\beta u(E\{c_2\})}_{\text{welfare under certainty}} > \underbrace{u(c_1)+\beta E\{u(c_2)\}}_{\text{welfare under uncertainty}}$$

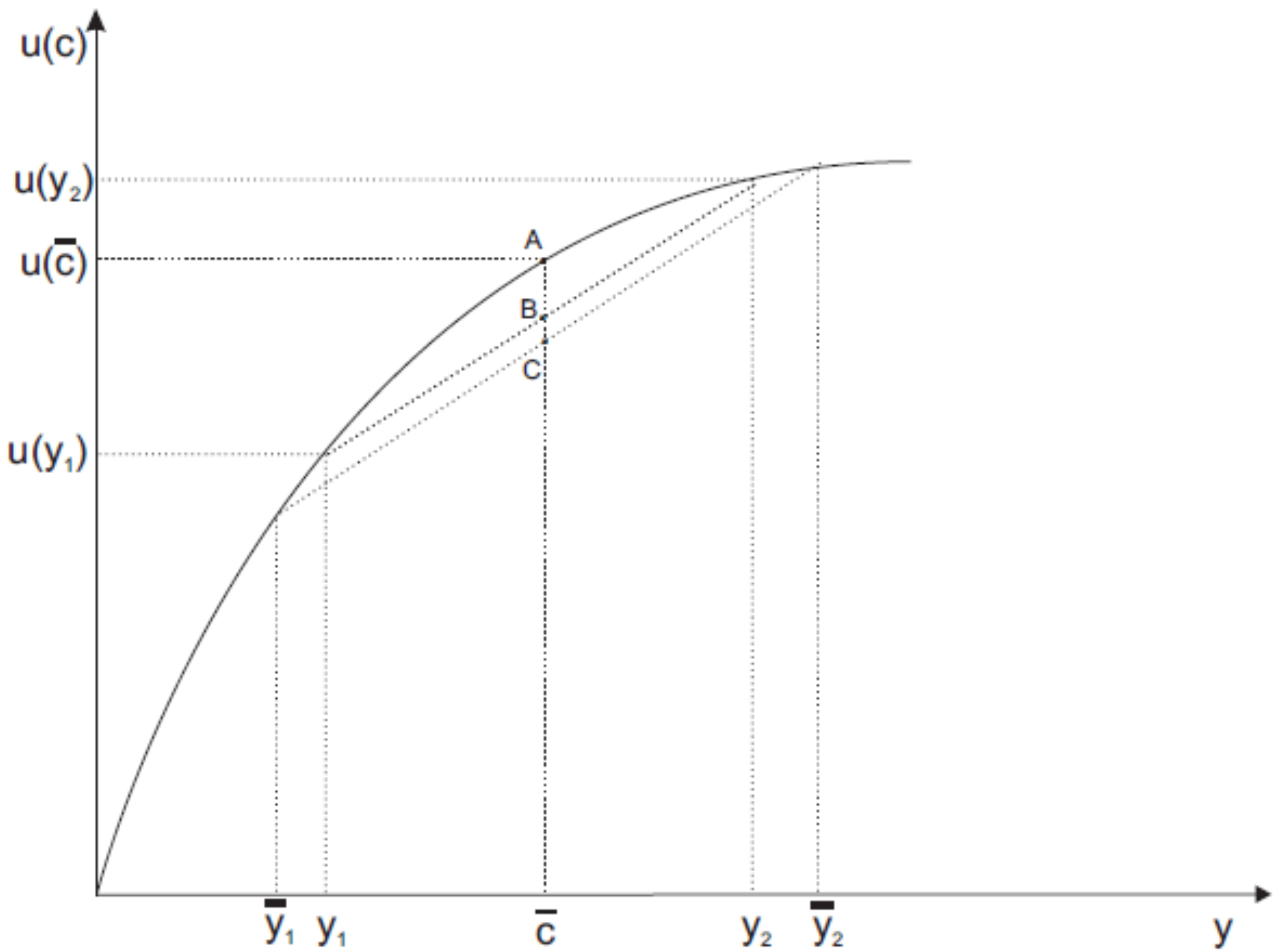
as

$$u(E\{c_2\}) > E\{u(c_2)\} = pu(c_2^H)+(1-p)u(c_2^L),$$

which holds in light of the strong concavity of $u(\cdot)$.

Same analysis of figure 1 applies to this case with :
point A denoting period - 2 utility under certainty and
point B denoting period - 2 utility under uncertainty.

Case 1 (cont.)



Case 1 (cont.)

Furthermore, it is easy to show that the more variable is output, the higher are the welfare costs. To see this consider $\tilde{y}_2^H \left(> y_2^H\right)$ and $\tilde{y}_2^L \left(< y_2^L\right)$ with the same expected value. By analogy with (34) and (35), we know that

$$\tilde{c}_2^H = \tilde{y}_2^H + \frac{1+r}{2+r}\left[y_1 - E\{y_2\}\right] \tag{39}$$

$$\tilde{c}_2^L = \tilde{y}_2^L + \frac{1+r}{2+r}\left[y_1 - E\{y_2\}\right] \tag{40}$$

Then, by strict concavity of $u(\cdot)$:

$$pu\left(\tilde{c}_2^H\right) + (1-p)u\left(\tilde{c}_2^L\right) < pu\left(c_2^H\right) + (1-p)u\left(c_2^L\right)$$

Still using figure 1, now the comparison is between point B and point C.

Case 2: precautionary savings $u'''(c)>0$

Given that under $u''' > 0$ $\rightarrow$ $E\{u'(c_2)\} > u'(E\{c_2\})$ and

from the Euler equation $u'(c_1) = E\{u'(c_2)\}$

then we have $u'(c_1) > u'(E\{c_2\})$

from where it follows that $c_1 < E\{c_2\}$ (41)

Therefore, unlike the previous case, consumers do not smooth consumption in an expected value sense. From (31) and (41) it follows that

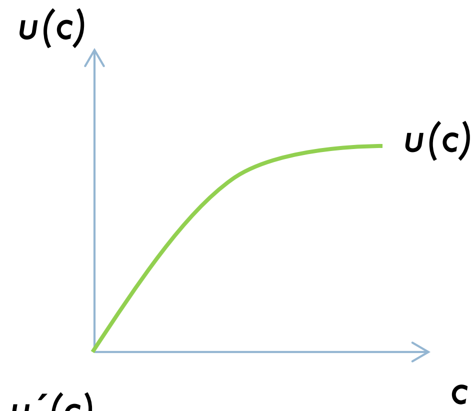
$$c_1 < \frac{1+r}{2+r}\left[y_1 + \frac{E\{y_2\}}{1+r}\right]$$

Consumption is thus lower than it would be in the certainty (or certainty equivalence) case as we have precautionary savings.

Case 2 (cont.)

Risk aversion : consumer dislike uncertainty

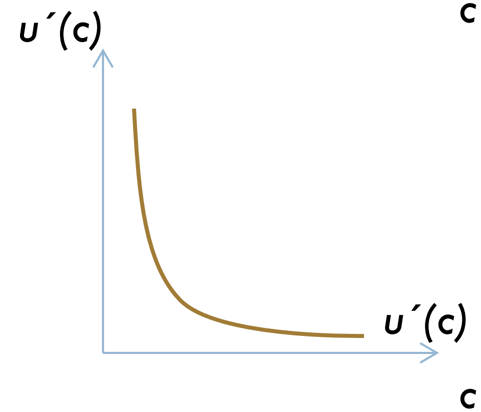
$\left(-u''/u'\right)$ (measured by the concavity of $u(\cdot)$)



Prudence : consumers are prepared to save

$\left(-u'''/u''\right)$ in anticipation of an uncertain outcome.

(measured by the convexity of $u'(\cdot)$)



Standard preferences - such as constant relative (or absolute) risk aversion - are characterized by $u''' > 0$ and will therefore exhibit precautionary saving.

Case 3: imprudence $u'''(c)<0$

Given that under $u''' < 0 \quad \rightarrow \quad E\{u'(c_2)\} < u'(E\{c_2\})$ and

from the Euler equation $u'(c_1) = E\{u'(c_2)\}$

then we have $u'(c_1) < u'(E\{c_2\})$

from where it follows that $c_1 > E\{c_2\}$ (42)

In this case, consumers are "imprudent" in the sense that they choose to dissave more than they would otherwise and will in fact consume more in period 1 than in the certainty case.

Therefore, the CA_1 deficit will be larger than in the certainty case.

It is still the case that consumption is not smooth over time and that is costly from a welfare point of view.

3.3 Complete Markets

Now households can buy contingent claims in international capital markets.

Households may buy a claim that promises to pay a unit of output in the good state of nature for the price $q^H/_{1+r}$ and a claim that promises to pay a unit of output in the bad state of nature for the price $q^L/_{1+r}$.

Given that these two assets span the possible states of nature $\{H,L\}$ they are enough to have complete markets.

Note that we could include more assets, but they will be redundant as a risk - free asset that pays one in any state of nature and costs $1/_{1+r}$.

3.3.1 Budget Constraints

Assume $b_0 = 0$. Denote by $b_1^j, j = H, L$ the number of claims purchased in period 1 that promise to pay one unit of output in the second period in state of nature j. The flow budget constraint for period 1 is thus:

$$\frac{q^H}{1+r} b_1^H + \frac{q^L}{1+r} b_1^L = y_1 - c_1 \qquad (43)$$

As in the incomplete markets case, the second-period flow budget constraint will depend on the realization of output. Then

$$c_2^H = b_1^H + y_2^H \qquad (44)$$

and

$$c_2^L = b_1^L + y_2^L \qquad (45)$$

Substituting (44) and (45) into (43), we obtain:

$$y_1 + \frac{q^H y_2^H + q^L y_2^L}{1+r} = c_1 + \frac{q^H c_2^H + q^L c_2^L}{1+r} \qquad (46)$$

3.3.2 Utility Maximization

Consumer chooses c_1, c_2^H, c_2^L to maximize (24) subject to (46):

$$\max_{\{c_1, c_2^H, c_2^L, \lambda\}} \quad u(c_1) + \beta\left[pu\left(c_2^H\right) + (1-p)u\left(c_2^L\right)\right] + \lambda\left[y_1 + \frac{q^H y_2^H + q^L y_2^L}{1+r} - c_1 - \frac{q^H c_2^H + q^L c_2^L}{1+r}\right]$$

The first-order necessary conditions for a maximum include:

$$\frac{\partial L}{\partial c_1} = u'(c_1) - \lambda = 0 \qquad (47)$$

$$\frac{\partial L}{\partial c_2^H} = \beta p u'\left(c_2^H\right) - \lambda \frac{q^H}{1+r} = 0 \qquad (48)$$

$$\frac{\partial L}{\partial c_2^L} = \beta (1-p) u'\left(c_2^L\right) - \lambda \frac{q^L}{1+r} = 0 \qquad (49)$$

Combining (48) and (49), we obtain:

$$\underbrace{\frac{pu'\left(c_2^H\right)}{(1-p)u'\left(c_2^L\right)}}_{\substack{\text{probability weighted relative} \\ u' \text{ across states of nature}}} = \underbrace{\frac{q^H}{q^L}}_{\substack{\text{relative price} \\ \text{of contingent claims}}} \qquad (50)$$

Utility Maximization (cont.)

If prices are actuarially fair, it is the case that : $\frac{q^H}{q^L} = \frac{p}{1-p}$ (51)

Imposing (51) into (50) it follows that $c_2^H = c_2^L$.

Hence, under actuarially fair prices, the consumer equates c_2^i.

To show that consumption will be smoothed across time, notice

that by arbitrage $\frac{1}{1+r} = \frac{q^H}{1+r} + \frac{q^L}{1+r} \Rightarrow 1 = q^H + q^L$

From (51) : $(1-p)q^H = pq^L = p(1-q^H) \Rightarrow q^H = p$ and $q^L = 1-p$
where the last term in the equality imposes the arbitrage condition.

Utility Maximization (cont.)

The first order conditions (47), (48) and (49) implies that :

$$\lambda = u'(c_1) = \frac{\beta(1+r)pu'(c_2^H)}{q^H} = \frac{\beta(1+r)(1-p)u'(c_2^L)}{q^L}$$

Using $q^H = p$ and $q^L = (1-p)$ and assuming $\beta(1+r)$ the above expression implies

$$\lambda = u'(c_1) = u'(c_2^H) = u'(c_2^L) \Rightarrow c_1 = c_2^H = c_2^L$$

Therefore, with complete markets, consumption is fully smoothed even when the output path is uncertain. $\Rightarrow \; corr(c_2, y_2) = 0$

4. No Commitment

- Suppose now that the country have access to international capital markets but it is not able to precommit itself to repaying its debt
- What are the implications of the absence of such precommitment?

Set-up of the Model

- Small open economy inhabited by a large number of identical, two-period-lived consumers.
- No uncertainty, perfect foresight.
- Only one good (tradable and non-storable).
- Small open economy: takes price of tradable good as given.
- The economy is endowed with a flow of the good (*i*.e. no production).
- No government.
- Perfect capital mobility in the sense that consumers can borrow/lend in international capital markets at a constant real interest rate, *r*.
- No precommitment from households to repay their debt.

4.1 Implications of No Precommitment

We have seen that under perfect capital mobility, precommitment and no uncertainty, the level of consumption in period $i = 1, 2$ is given by :

$$c_i = c = \frac{1+r}{2+r}\left(y_1 + \frac{y_2}{1+r}\right) \tag{52}$$

Assuming $y_1 < y_2$ and substituting (52) into $b_1 = y_1 - c_1$, then

$$b_1 = \frac{y_1 - y_2}{2+r} < 0 \tag{53}$$

Letting d_1 denote net debt *(i.e.* $d_1 \equiv -b_1$*)*, under precommitment households would consume $c_2 = y_2 - (1+r)d_1 < y_2$ in period 2. But, in the absence of precommitment the country would prefer not to repay its debt and consume $c_2 = y_2$. Rational creditors would anticipate such incentives and not lend to this economy in period 1

$\therefore$ The economy would thus be in a state of financial autarky

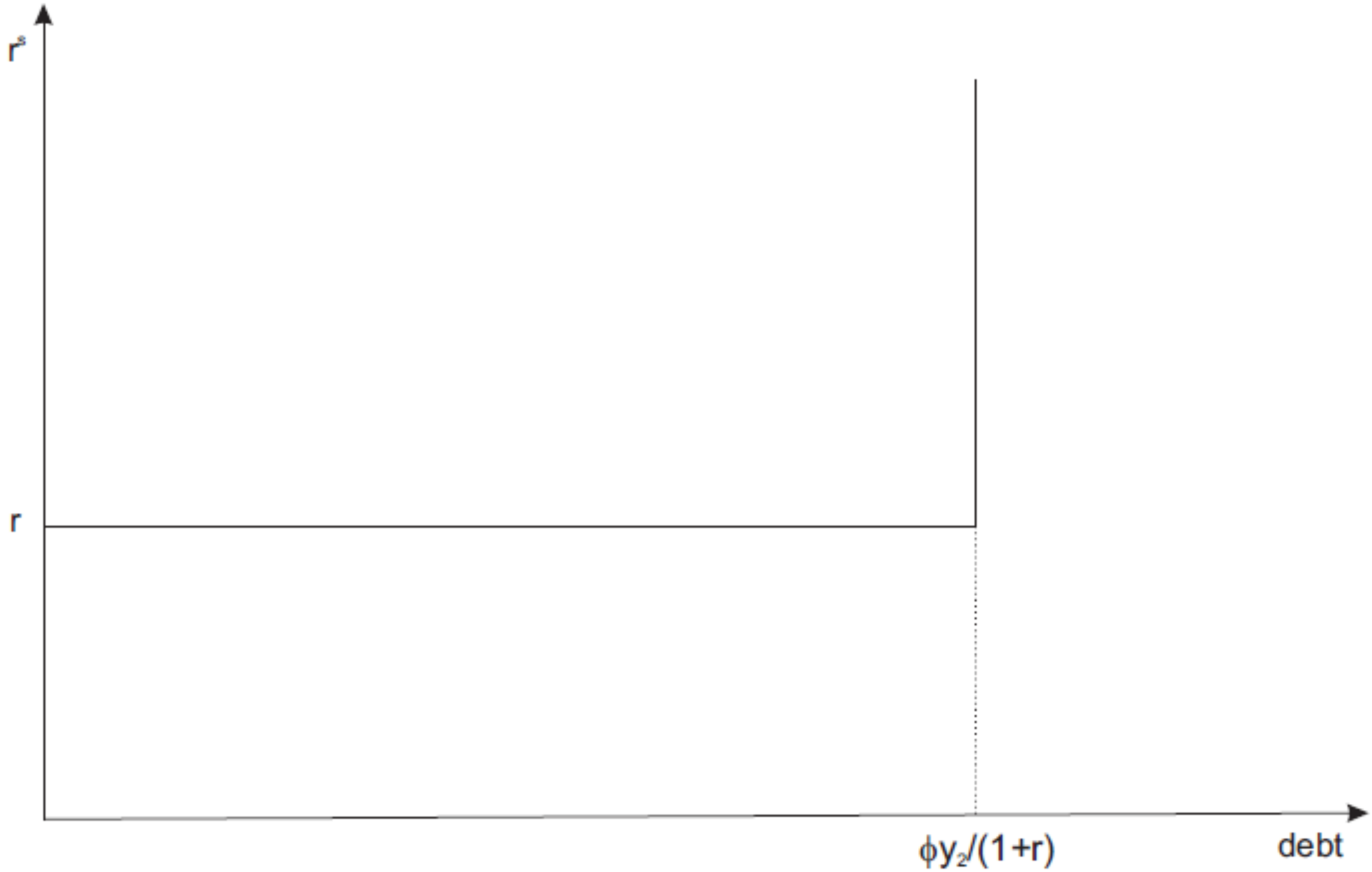
4.2 Cost of Default

Now suppose that there is some exogenous cost ϕy_2 of defaulting in period 2, with $\phi \in [0,1)$. In this case the country will repay its debt if and only if consumption under no default is higher than consumption under default :

$$y_2 - (1+r)d_1 \geq (1-\phi)y_2 \quad \Rightarrow \quad \mathrm{d}_1 \leq \frac{\phi y_2}{1+r} \tag{54}$$

$\therefore$ This no - default condition implies an upper bound on the feasible amount of borrowing in the first period : Because households will choose to default when debt is higher than $\frac{\phi y_2}{1+r}$, creditors will not lend over and above that quantity

4.1 Costs of Default (cont.)



4.2 Costs of Default (cont.)

Formally, the consumer´s maximization problem becomes

$$\max_{\{d_1\}} W = u(y_1 + d_1) + \beta u(y_2 - (1+r)d_1)$$

subject to

$$y_1 + \frac{y_2}{1+r} = c_1 + \frac{c_2}{1+r}$$

and

$$d_1 \leq \frac{\phi y_2}{1+r}$$

where we have used the flow constraints

$$c_1 = y_1 + d_1 \quad (55)$$

$$c_2 = y_2 - (1+r)d_1 \quad (56)$$

4.2 Costs of Default (cont.)

$$\max_{d_1} L = u(y_1 + d_1) + \beta u[y_2 - (1+r)d_1] + \chi\left(\frac{\varphi y_2}{1+r} - d_1\right)$$

The Kuhn - Tucker conditions :

$$u'(c_1) = u'(c_2) + \chi$$

$$\frac{\phi y_2}{1+r} - d_1 \geq 0,\ \chi\left(\frac{\phi y_2}{1+r} - d_1\right) = 0 \qquad (57)$$

Thus, there are two possible solutions to this problem

4.2.1 Nonbinding Constraint

If y_1 and y_2 are such that (54) is satisfied, then from (57)

$$\left.\begin{array}{c} \chi = 0 \\ and \\ u'(c_1) = u'(c_2) \end{array}\right\} \therefore \text{Full comsumption smoothing}$$

4.2.2 Binding Constraint

Suppose that y_1 and y_2 are such that the economy would like to borrow in the first period more than what international creditors are willing to lend. Then $\chi > 0$ and, from (57):

$$u'(c_1) > u'(c_2) \Rightarrow c_1 < c_2$$

From (55) and (56), consumption will be:

$$c_1 = y_1 + d_1^{max} \quad (58)$$

$$c_2 = y_2 - (1+r)d_1^{max} \quad (59)$$

where $d_1^{max} = \frac{\phi y_2}{1+r}$ is the maximum amount of borrowing

Hence, consumer cannot smooth consumption

4.3 Default Risk under Uncertainty

We now introduce uncertainty into the model (i.e., an uncertain level of period 2 output)

4.3.1 Supply of International Credit

Assume that y_2 is a random variable that has a density function

$$f(y_2) = \begin{cases} \dfrac{1}{y_2^H}, & 0 \le y_2 \le y_2^H, \\ 0, & otherwise \end{cases} \qquad (60)$$

and a cumulative distribution function

$$f(\alpha) = \begin{cases} 0, & \alpha \le 0, \\ \dfrac{\alpha}{y_2^H}, & 0 < \alpha < y_H^2, \\ 1, & \alpha \ge y_2^H \end{cases} \qquad (61)$$

4.3.1 Supply of International Credit (cont.)

We assume that the country will repay its debt only if

$$d_1(1+r^s) < \phi y_2$$

Thus, the default decision is given by

$$\text{If } y_2 \leq y_2^* \quad \Rightarrow \quad \text{Default}$$

$$\text{If } y_2 > y_2^* \quad \Rightarrow \quad \text{No default}$$

where $y_2^* \equiv \dfrac{d_1(1+r^s)}{\phi}$ is the default threshold (62)

Using (61) and (62), the probability of default (denoted by π) is given by

$$\pi = \Pr(y_2 < y_2^*) = F\left[\frac{d_1(1+r^s)}{\phi}\right] \qquad (63)$$

4.3.1 Supply of International Credit (cont.)

Hence,

1) For high levels of debt $\left(y_2^* \geq y_2^H\right) \Rightarrow$ the economy will always default

2) For low levels of debt $\left(d_1 \leq 0\right) \Rightarrow$ the economy will never default

3) For intermediate levels of debt $(0 < y_2^* < y_2^H) \Rightarrow$ the probability of default is given by

$$\pi = \frac{y_2^*}{y_2^H} = \frac{\left(1+r^s\right)d_1}{\phi y_2^H} \tag{64}$$

How does π vary with d_1? Suppose that international lenders are risk neutral and has an opportunity cost of r. Then,

$$1 + r = \left(1 - \pi\right)\left(1 + r^s\right) \tag{65}$$

Solve for $\left(1 + r^s\right)$ from (65), substitute into (64) and totally differentiate :

$$\frac{d\pi}{dd_1} = \frac{1+r}{\phi y_2^H\left(1-2\pi\right)} > 0 \text{ in equilibrium} \tag{66}$$

4.3.1 Supply of International Credit (cont.)

To derive an explicit functional form for the supply of funds; substitute (64) into (63) and rearrange

$$\mathrm{r}^{s} = \frac{1+r}{1-\left(1+r^{s}\right)d_1/\phi y_2^H} - 1 \qquad (67)$$

Solving explicitly for r^s :

$$r^{s} = \begin{cases} r, & \mathrm{d}_1 \leq 0 \\ \dfrac{2(1+r)d_1^{\max}}{d_1}\left(1-\sqrt{1-\dfrac{d_1}{d_1^{\max}}}\right)-1, & 0 < \mathrm{d}_1 \leq d_1^{\max} \end{cases} \qquad (68)$$

where $\mathrm{d}_1^{\max} \equiv \dfrac{\phi y_2^H}{4(1+r)}$ is the maximum amount of debt that can be sustained by international markets.

4.3.1 Supply of International Credit (cont.)

Derive (68) with respect to d_1 :

$$\frac{dr^s}{dd_1} = \frac{(1+r)d_1^{\max}}{d_1^2\sqrt{1-\frac{d_1}{d_1^{\max}}}}\left(2-\frac{d_1}{d_1^{\max}}-2\sqrt{1-\frac{d_1}{d_1^{\max}}}\right) > 0, \qquad 0 < d_1 \le d_1^{\max}$$

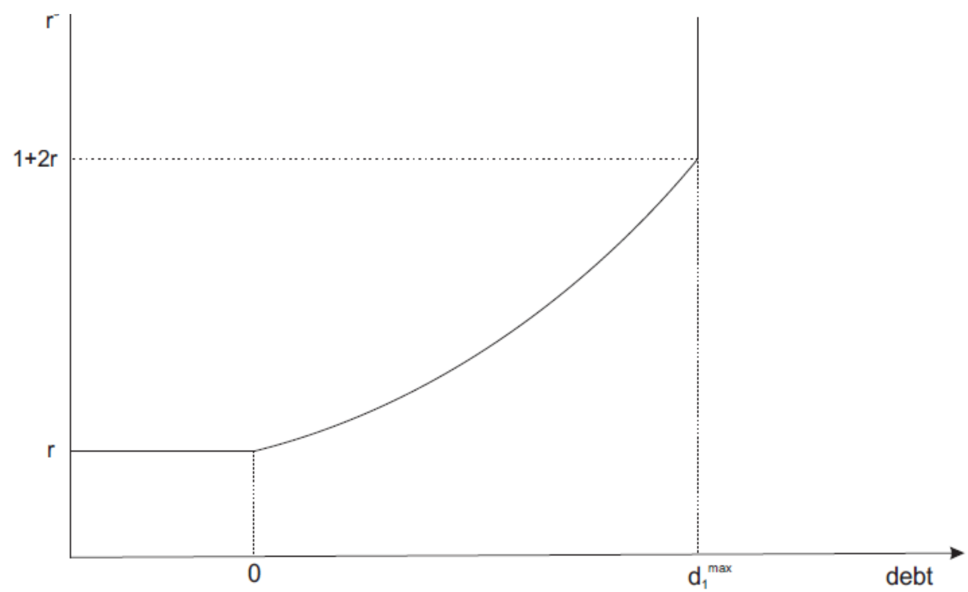
And evaluate (68) at $d_1 = d_1^{\max}$

to find the maximum interest rate r^s

$$r^s\Big|_{d_1=d_1^{\max}} = 1+2r \qquad (69)$$

$\therefore$ r^s is an increasing function of d_1, reflecting that the higher the level of debt the more likely that the economy will default

Figure 3. Upward sloping supply of funds



4.3.2 Demand for International Credit

Assume that the lifetime utility is given by

$$W = c_1 + \frac{1}{1+\delta} \mathrm{E}\{c_2\} \tag{70}$$

To introduce a motive for borrowing, suppose that $\delta > r$

where $\delta (> 0)$ is the discount rate and r is the risk - free world interest rate

Assuming $b_0 = 0$, then

$$c_1 = y_1 + d_1, \tag{71}$$

$$c_2^D = (1-\phi) y_2, \tag{72}$$

$$c_2^{ND} = y_2 - (1 + r^s) d_1 \tag{73}$$

where c_2^D and c_2^{ND} denote concumption in period 2 under default and no default

4.3.2 Demand for International Credit (cont.)

To express the lifetime utility function (70) as a function of d_1, note that by the law of iterated expectations

$$E\{c_2\} = \pi\, E\{c_2^D\} + (1-\pi) E\{c_2^{ND}\} \qquad (74)$$

Substituting (74) into lifetime utility (70), then

$$\mathrm{W} = \mathrm{c}_1 + \frac{\pi}{1+\delta} E\{c_2^D\} + \frac{1-\pi}{1+\delta} \phi E\{c_2^{ND}\} \qquad (75)$$

Taking the conditional expectation of (72) and (73), substituting the results and (71) into (75), and using that $1+r = (1\text{-}\pi)(1+r^s)$ we have

$$\mathrm{W} = y_1 + \frac{1}{1+\delta} E\{y_2\} + \frac{\delta - r}{1+\delta} d_1 - \frac{\pi}{1+\delta} \phi E\{y_2 | D\} \qquad (76)$$

4.3.2 Demand for International Credit (cont.)

Now notice that conditional on the economy having defaulted, y_2 varies uniformly in $[0, y_2^*]$. And conditional on the economy not having defaulted, y_2^* varies in $(y_2^*, y_2^H]$. Thus :

$$E\{y_2 | D\} = \frac{y_2^*}{2} = \frac{\pi y_2^H}{2}, \tag{77}$$

and

$$E\{y_2 | ND\} = \frac{y_2^* + y_2^H}{2} = \frac{(1+\pi) y_2^H}{2} \tag{78}$$

which follows from $\pi = y_2^* / y_2^H$

Substituting (77) into (76), we obtain the consumer´s problem in terms of d_1

$$\max_{d_1} \; W = y_1 + \frac{1}{1+\delta} E\{y_2\} + \frac{\delta - r}{1+\delta} d_1 - \frac{\phi \pi^2 y_2^H}{2(1+\delta)} \tag{79}$$

4.3.2 Demand for International Credit (cont.)

F.O.C :

$$\frac{\delta - r}{1+\delta} - \frac{2\phi\pi y_2^H}{2(1+\delta)}\frac{d\pi}{dd_1} = 0 \;\Rightarrow\; \delta - r = \phi\pi y_2^H \qquad (80)$$

In equilibrium :

1) The marginal benefit of borrowing to increase period´s 1 consumption(LHS) must equal the marginal cost of borrowing

2) Since $\delta > \mathrm{r}$ (by assumption) $\Rightarrow \left(\mathrm{d}\pi/\mathrm{dd}_1\right) > 0$ in equilibrium

Borrowing one additional unit increases the marginal cost on two accounts :

1) For a given loss $\phi E\{y_2 | D\}$, increases the probability of default by $d\pi = dd_1$

2) For a given π, increases the expected loss in case of default by $\phi\left[d\left(E\{y_2 | D\}\right)/d\pi\right] = \phi y_2^H$, which occurs with probability π

4.3.2 Demand for International Credit (cont.)

Using (64) - (66) and (80) to compute the equilibrium solution :

$$\pi = \frac{\delta - r}{1 + 2\delta - r} \qquad (81)$$

$$d_1 = \frac{\phi y_2^H (1+\delta)(\delta - r)}{(1+r)(1+2\delta - r)^2} \qquad (82)$$

$$r^s = \frac{1+r}{1+\delta}(1 + 2\delta - r) - 1 \qquad (83)$$

Hence, in equilibrium

1) The cost of default (ϕ) do not affect the probability of default (π)

2) A more impatient economy (with a higher δ) will borrow more, pay a higher real interest rate and face a higher probability of default.

4.3.2 Demand for International Credit (cont.)

By substituting (81) and (82) into (79) we find the equilibrium level of welfare:

$$W = y_1 + \frac{1}{1+\delta} E\{y_2\} + \phi \frac{y_2^H (\delta - r)^2}{2(1+2\delta - r)(1+r)(1+\delta)} \qquad (81)$$

$\therefore$ Higher costs of default are welfare improving

If there were no sovereing risk, this economy would want to consume all its period 2 endowment in period 1, i.e.

$$d_1\Big|_{no\ sovereign\ risk} = \frac{E\{y_2\}}{1+r}$$

But with sovereign risk the maximum amount of borrowing is given by:

$$d_1^{\max} = \frac{\phi}{2} \frac{E\{y_2\}}{1+r}$$

$\therefore$ Welfare under sovereign risk (even for $\phi = 1$) is lower than welfare under no sovereign risk

4.3.3 Upward-Sloping Supply of Funds

An upward sloping supply of funds has important consequences for the optimal choice of borrowing by a small open economy :

1) The economy will choose not to smooth consumption : Given the imperfection in international capital markets, it is optimal not to smooth consumption over time since more borrowing increases borrowing costs.

2) The social marginal cost of borrowing is higher than the private marginal cost. It would thus be optimal to impose a tax on foreign borrowing that would reduce private borrowing to the socially optimal level (a second - best policy).

Problem 1: financial autarky

- Financial autarky: Consider a small open economy where first-period endowment is $y_1=10$ and second-period is $y_2=14$, both known with certainty. Assume that preferences are logarithmic with a lifetime utility given by:

$$W=ln(c_1)+0.8ln(c_2)$$

(a) Assuming that the country has access to international capital markets at an interest rate $r = 0.1$, find c_1, c_2, and b_1. What is the pattern of consumption and how is this related to the difference between β and $(1/(1+r))$? Compute the welfare level associated to these consumption levels.

(b) Now assume that the economy does not have access to international capital markets. In this context, derive the levels of c_1, c_2, and b_1. What is the level of domestic interest rates ρ? Compute the welfare level associated to these consumption levels. Compare this number with the case of perfect capital markets and discuss the welfare implications of having limited access to international capital markets.

Problem 2: incomplete markets

- Uncertain output path: Consider the model described in chapter 2, section 3 where the path of output is uncertain. The first-period endowment is $y_1 = 10$ with certainty. The second-period endowment is stochastic:

$$y_2 = \begin{cases} 18 & \text{with probability } \frac{2}{3} \\ 6 & \text{with probability } \frac{1}{3} \end{cases}$$

- Assume that the world interest rate $r = 0.1$ and that $\beta = (1/(1+r))$.

(a) In the case of incomplete capital markets and certainty equivalence solve for c_1, b_1, c_2^H, c_2^L, $E\{c_2\}$, $Var\{c_2\}$, and $Corr(c_2, y_2)$.

(b) Specify a linear utility function and compare the welfare under certainty and under uncertainty.

$$W = c_1 + \beta E\{c_2\}$$

(c) Specify a quadratic utility function and compare the welfare under certainty and under uncertainty.

$$W = \left(c_1 - \frac{c_1^2}{2}\right) + \beta E\left\{c_2 - \frac{c_2^2}{2}\right\}$$

(d) Now assume that there are complete asset markets in the sense that households can buy contingent claims in international capital markets as in section 3.3. Under the assumption of actuarially fair prices, find the relative price of $\frac{q^H}{q^L}$.

(e) You can normalize the price $q^L = 1$. Using this find the consumption levels c_1, c_2^H, c_2^L, then find the amount of securities b_1^H, and b_1^L that the household buys, finally find the variance of second-period consumption $Var\{c_2\}$, and the second-period correlation between output and consumption $Corr(c_2, y_2)$.

Problem 3: No commitment

- An ad-hoc upward sloping supply of funds: Let preferences be given by:

$$W = c_1 + \frac{c_2}{\delta}$$

where δ is the discount rate, which by assumption is above the world real interest rate, r.

- The flow constraints are given by:

$$c_1 = d_1$$
$$c_2 = y_2 - (1 + r^s)d_1$$

where d_1 is net external debt and $y_2 > 0$ is second period's output.

- The economy faces an upward sloping supply of funds of the form

$$r^s = r + \alpha d_1$$

with $\alpha > 0$ capturing the risk-premium associated with a given level of indebtnedness.

(a) Solve the planner's problem. Derive a reduced-form solution for the equilibrium values of d_1 and r^s.

(b) Solve the free market's problem. Derive a reduced-form solution for the equilibrium values of d_1 and r^s.

(c) Discuss the differences in the levels of consumption, debt and interest rates between both solutions.

(d) Show that a borrowing tax rate $\tau = \alpha d_1$ in the free market economy, that will be charged in the second period such that the market economy will be:

$$c_2 = y_2 - (1 + r^s)d_1 - \tau d_1$$

could replicate the central planner's solution.

Solution Problem 1

Q1					
Perfect Financial Markets				Financial Autarky	
y1	10				
y2	14				
r	0.1				
beta	0.8				
c1	12.62626			c1	10
c2	11.11111			c2	14
b1	-2.62626			b1	0
				rho	0.75
W	1.937881			W	1.916902

Solution Problem 2

Q2													
Uncertainty									Complete Markets				
y1	10		y2H	18		pH	0.667		qH	2			
E(y2)	14		y2L	6		pL	0.333		qL	1			
r	0.1												
beta	0.909090909												
c1	11.9047619		c2H	15.90476					c1	11.90476		c2H	11.90476
E(c2)	11.9047619		c2L	3.904762		11.90476			E(c2)	11.90476		c2L	11.90476
b1	-1.904761905					11.90476			b1H	-6.09524	-6.09524		
									b1L	5.904762	5.904762		
Var y2	32								Var y2	32			
Var c2	32								Var c2	0			
Corr(c2,y2)	1	0							Corr(c2,y2)	0			
W Linear utility		22.72727		W = c1 + beta*c2									
W certainty quadratic		-112.554		W = (c1 - 0.5*c1^2) + beta*(E(c2) - 0.5*E(c2^2))									
W uncertainty quadratic		-127.1		W = (c1 - 0.5*c1^2) + beta*(pH*(c2H - 0.5*c2H^2) + pL*(c2H - 0.5*c2H^2))									